Tutorial 7 for MATH 2020A (2024 Fall)

1. For the following vector fields defined on their natural domain, use either **the component test** or **the curl test** to determine whether they are conservative or not:

   (a) $\mathbf{F}(x,y) = y\mathbf{i} + (x+z)\mathbf{j} - y\mathbf{k}$;

   (b) $\mathbf{F}(x,y) = (z+y)\mathbf{i} + z\mathbf{j} + (y+x)\mathbf{k}$.

   Solution: (a) not conservative; (b) not conservative

2. For the following vector field $\mathbf{F}$ defined on its natural domain,

$$\mathbf{F}(x,y,z) = \left(\ln x + \sec^2(x+y)\right)\mathbf{i} + \left(\sec^2(x+y) + \frac{y}{y^2+z^2}\right)\mathbf{j} + \frac{z}{y^2+z^2}\mathbf{k},$$

   (a) Determine its natural domain Ω;

   (b) Determine whether Ω is open, connected or simply connected;

   (c) Consider the same $\mathbf{F}$ but is defined in the cube $R = \{(x,y,z) : 0 < x,y,z < \frac{\pi}{4}\}$, show $\mathbf{F}$ on R is conservative and find one explicit potential function $\phi(x,y,z)$ of $\mathbf{F}$.

   Solution: (a) $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x > 0, y^2+z^2 \neq 0, x+y \neq \frac{\pi}{2} + k\pi (k \in \mathbb{Z})\}$.

   (b) Ω is open, but is neither connected nor simply connected.

   (c) The pontential family is $\phi(x,y,z) = x\ln x - x + \tan(x+y) + \frac{1}{2}\ln(y^2+z^2) + C$.

3. Evaluate the following work done integrals of conservative vector fields by using their potential functions:

   (a)

$$\int_{(1,1,1)}^{(1,2,3)} 3x^2\,\mathrm{d}x + \frac{z^2}{y}\,\mathrm{d}y + 2z\ln y\,\mathrm{d}z,$$

   (b)

$$\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}\,\mathrm{d}x + \left(\frac{1}{z} - \frac{x}{y^2}\right)\mathrm{d}y - \frac{y}{z^2}\,\mathrm{d}z.$$

   Solution: (a)$9\ln 2$; (b)0.